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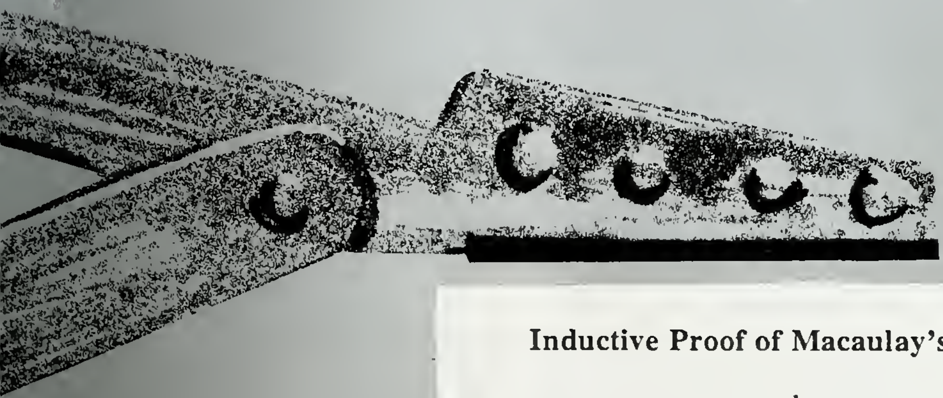

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Thomas Dubé

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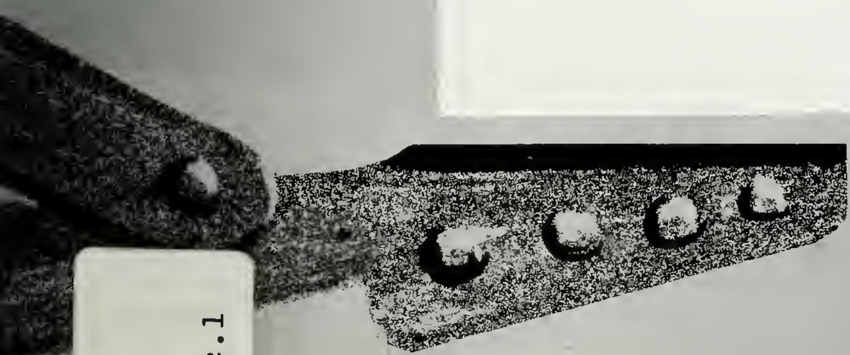

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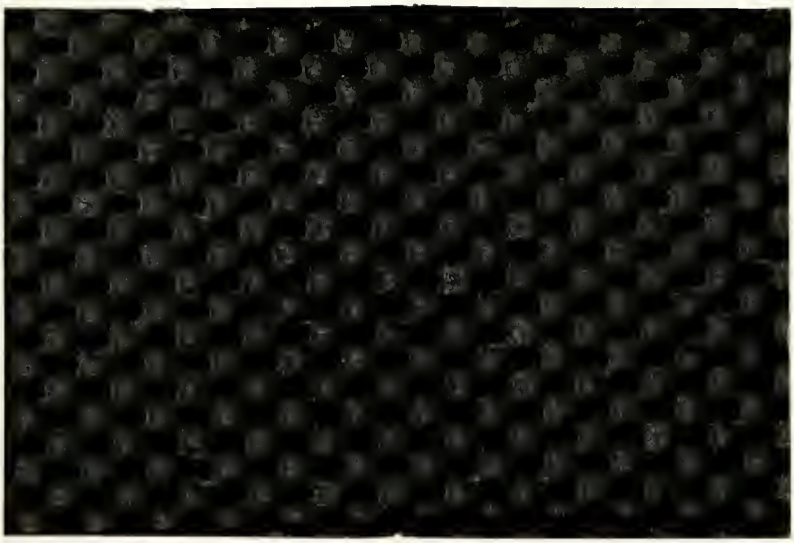

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Inductive Proof of Macaulay's Theorem

Thomas W. Dubé

June 9, 1989

Abstract

Macaulay's theorem states that for every ideal I in a multi-variate polynomial ring $A = K[x_1, \ldots, x_n]$ there is a lexicographic monomial ideal L_I such that L_I has the same Hilbert function as I. This theorem allows one to characterize exactly the form of Hilbert polynomials.

This report uses a recursive characterization of a Borel-fixed monomial ideal to provide a new inductive proof of Macaulay's famous theorem.

1 Introduction

A power product $A = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is lexicographically greater than $B = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ if the first exponent at which they differ has $a_i > b_i$. This relationship is written as $A \underset{L}{>} B$ and defines a total ordering on the set of power products. The word <u>degree</u> will be used to refer to the total-degree of a power product, i.e.

$$\deg(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = \sum_{i=1}^{n} a_i .$$

A monomial ideal L is called a <u>lexicographic ideal</u> if for every $B \in L$ all the power products A with $\deg(A) = \deg(B)$ and $A \underset{L}{>} B$ are also in the ideal L.

Theorem 1 (Macaulay) *For any homogeneous polynomial ideal I there exists a lexicographic ideal L such that I and L share the same Hilbert function.*

This theorem was first proved in the classic paper by Macaulay [Ma 27]. As a corollary of this theorem, Macaulay was able to characterize the class of polynomials which may occur as the Hilbert polynomials of ideals.

Corollary 2 *Let I be a non-zero ideal of $K[x_1, \ldots, x_n]$. Then the Hilbert polynomial of I can be expressed uniquely in the form*

$$\overline{\varphi}_I(d) \;=\; \sum_{i=0}^{n-2} \binom{d+i-m_i}{i+1} + 1$$

where $m_0 \geq m_1 \geq m_2 \geq \cdots \geq m_{n-2} \geq 0$.
Conversely, for every polynomial $f(z)$ which has this form, there exists a lexicographic ideal L such that $\overline{\varphi}_L(z) = f(z)$.

This corollary can be proved relatively easily by considering the Hilbert polynomials of lexicographic ideals. The Hilbert polynomial of a lexicographic ideal is completely defined by considering the $\underset{L}{>}$-last power product in the ideal. If this power product is $A = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, then the Hilbert polynomial has the form given above with

$$m_i \;=\; \sum_{k=1}^{n-i-1} a_i \; .$$

Macaulay's theorem is more powerful than the corollary in that the theorem states a property of the Hilbert function, while the corollary only considers the Hilbert polynomial which is valid at large degrees. Macaulay prefaced the original proof of his theorem with the warning:

> It is too long and complicated to provide any but the most tedious reading.

The theorem has been often reproved, and the proofs are becoming shorter and less tedious. Clements and Lindström [ClLi 69] for example provides a very readable proof. However, the subtlety of the theorem continues to thwart those who attempt concise proofs. As recently as 1982, after providing a proof of the theorem in his thesis, Bayer comments:

> It would seem that such an obvious statement would lend itself to a simpler proof, but after rejecting many false statements, this author was unable to find one.

This report comprises yet another proof of Macaulay's theorem. Although it is hoped that the new proof is no worse than the ones already in the literature, there is no claim that it is any better. The proof is provided primarily as a further demonstration of the use of cone decompositions. Once again, a decomposition of the ideal allows an inductive proof of a constructive nature. The particular characterization of Borel fixed ideals used in the proof may also be employed in other problems concerning these ideals.

2 Background

2.1 Lexicographic Ideals

Recall the definition of a lexicographic ideal L: If a degree d power product t is in L, then every degree d power product s which is lexicographically greater than t is also in L.
Notation: For a lexicographic ideal L, let ℓ_d denote the *last* degree d power product in L. If L contains no power products of degree d than ℓ_d is defined to be 0.

The ideal L is fully defined by listing the set

$$\{\ell_0, \ell_1, \ell_2, \ldots\} ,$$

since any power product p is in L if and only if $p \underset{L}{\geq} \ell_{\deg(p)}$.

Lemma 3 *For any power product $t = x_1^{a_1} \cdots x_n^{a_n}$, the set of power products $\{s : \deg(s) \leq \deg(t) \ \& \ s \underset{L}{\geq} t\}$ generates a lexicographic ideal L whose power products are $\{p : p \underset{L}{\geq} t\}$.*

Proof. Let I and L denote the following ideals:

$$\begin{aligned} I &= (\{s : \deg(s) \leq \deg(t) \ \& \ s \underset{L}{\geq} t\}) \\ L &= (\{p : p \underset{L}{\geq} t\}). \end{aligned}$$

Then L is a lexicographic ideal with ℓ_d the lexicographically last degree d power products satisfying $\ell_d \underset{L}{\geq} t$.

$L \subseteq I$: For $p = x_1^{b_1} \cdots x_n^{b_n} \in L$, let j be the first index such that $a_j \neq b_j$. Since $p \underset{L}{>} t$ it follows be the definition of the lexicographic ordering that $b_j > a_j$. Let $t_j = x_1^{a_1} \cdots x_j^{a_j}$ be the projection of t onto the first j variables. There are two cases to consider:

1. $t_j = t$. In this case p is a multiple of t and hence in I.
2. $\deg(t_j) < deg(t)$. Then $s = x_j t_j$ is in the basis for I, and p is a multiple of s.

$I \subseteq L$: Assume otherwise and let p be a power product in $I - L$. Since $p \notin L$, it follows that $p \underset{L}{<} t$. In the basis of I, every power product s satisfies $s \underset{L}{\geq} t$. Now since $p \in I$, $p = as$ where s is a basis element of I. But this leads to the contradiction:

$$p \underset{L}{<} t \underset{L}{\leq} s \underset{L}{\leq} as = p \ .$$

□

The ideal L described by this lemma will be called the lexicographic ideal defined by t.

2.2 Borel Fixed Ideals

Definition: A monomial ideal I is said to be Borel-fixed if it is invariant under an upper-trangular linear change of coordinates. In practice, it is often convenient to use one of the equivalent definitions:

1. A monomial ideal I is Borel-fixed if for each pair of variables x_i. x_j such that $i < j$,

$$x_j M \in I \quad \text{implies} \quad x_i M \in I \ .$$

2. A monomial ideal I is Borel-fixed if

$$x_{i+1} M \in I \quad \text{implies} \quad x_i M \in I \ .$$

This can be sharpened to provide the following effective criteria for a Borel-fixed ideal.

Lemma 4 *Let I be a monomial ideal generated by a set of monomials $F = \{f_1, \ldots, f_r\}$, and let d be the maximum of the degrees of the f_i. Then, I is Borel fixed if and only if for each monomial M with* $\deg(M) \leq d$

$$x_{i+1}M \in I \quad \textit{implies} \quad x_i M \in I \ .$$

Proof. The implication in the $\Longrightarrow$ direction follows from the definition.

Conversely, assume that for $\deg(M) < d$, $x_{i+1}M \in I$ implies $x_i M \in I$. For any i and monomial M such that $x_{i+1}M \in I$, it must be shown that $x_i M \in I$. Since F is a monomial basis for I, $x_{i+1}M = f_j P$ for some f_j and monomial P. If the exponent of x_{i+1} in P is nonzero, then $M \in I$, and hence $x_i M \in I$. Otherwise, $f_j = x_{i+1}N$ for some N, then $x_i N \in I$ and therefore $x_i M = x_i NP \in I$.
□

In [Ba 82], [Gi 84] a key step in the analysis of Grōbner basis complexity is the transformation to generic coordinates. In this usage, *generic coordinates* for an ideal I refers to a coordinate system where the head ideal of I w.r.t the reverse lexicographic ordering is Borel-fixed. The term *generic* is used because after nearly any random change of coordinates this condition will occur.

[Ba 82] provides an informal proof that if the coefficient field K is infinite, then almost all linear upper triangular coordinate changes result in a transformation to generic coordinates. This same conclusion follows from [Ga 73], but there analytical methods are employed. [Du 88a] provides a formal construction which shows that if the coefficient field is infinite, than a generic coordinate system can always be found.

3 Borel Fixed Sets

This section is used to establish several properties of sets which are Borel fixed. A new recursive characterization of Borel fixed sets will also be presented.

We begin by listing three simple properties of Borel fixed sets. These properties are all easily verifiable, and the proofs will be omitted. Let B be an arbitrary Borel fixed set. Then,

1. For $j = 1, \ldots, n$, $B \cap \mathrm{PP}[x_1, \ldots, x_j]$ is Borel fixed.

2. For any degree d, let g be the lexicographically last power product in B. Then $B - \{g\}$ is Borel fixed.

3. For any degree d, let h be the lexicographically first power product not in B. Then $B \cup \{h\}$ is Borel fixed.

Now, for any set B of degree $d > 0$ power products ($d \geq 1$) partition B into the sets $R_1(B), R_2(B), \ldots, R_n(B)$ where $b \in B$ is in the subset $R_i(B)$ if x_i is the last variable which has a non-zero exponent in b. In other words,

$$R_i(B) = \{x_1^{a_1} \cdots x_i^{a_i} \in B : a_i > 0\} .$$

Since every power product in the set $R_i(B)$ has a non-zero exponent of x_i, the set $R_i(B)$ can be put in a one-to-one correspondence with the set

$$S_i(B) = \{x_i^{-1}h : h \in R_i(B)\} .$$

And so,

$$B = \oplus_{i=1}^{n}\{x_i h : h \in S_i(B)\} .$$

The definition of the sets $S_i(B)$ now allows a powerful recursive characterization of a Borel fixed set.

Lemma 5 *A set $B \in \mathrm{PP}[x_1, \ldots, x_n]$ of degree d power products is Borel fixed if and only if*

1. *$B \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]$ is Borel fixed.*
2. *$S_n(B)$ is Borel fixed.*
3. *$S_n(B) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}] \subseteq S_{n-1}(B)$.*

The simple proof of this lemma will follow this paragraph, but first the statement of this lemma deserves a bit of discussion. Any Borel fixed set of power products B can be split into those which do not include the variable x_n (i.e. $B \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]$), and those which do ($R_n(B)$). The first 2 conditions of this lemma constrains these two sets. The third condition describes how these two sets must fit together. This is a useful definition for inductive proofs, because the two sets are in a sense simpler than B itself. The set $B \cap \mathrm{PP}[x_1, \ldots, x_n]$ has only $n-1$ variables, and the set $S_n(B)$ is of degree $d-1$.

Proof. If B is Borel, then clearly these three properties must hold. Suppose conversely that B is not Borel. Then there exists an h and an i such that $x_{i+1}h \in B$, but $x_i h \notin B$. There are three cases to consider.

1. If $x_{i+1}h \in \mathrm{PP}[x_1, \ldots, x_{n-1}]$ then $B \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]$ is not Borel.

2. If h is a multiple of x_n, then $S_n(B)$ is not Borel.

3. If h is not a multiple of x_n, and $i = n-1$, then
   $x_n h \in R_n(B) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]$, but $x_{n-1}h \notin R_{n-1}(B)$.
   Hence $h \in S_n(B) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}] - S_{n-1}(B)$.

□

In applying this lemma, the cardinality of the sets will be considered. Note that

$$|B \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]| = \sum_{i=1}^{n-1} |S_i(B)| ,$$

and similarly,

$$|S_n(B) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]| = \sum_{i=1}^{n-1} |S_i(S_n(B))| .$$

A nice property of Borel-fixed monomial ideals is that they admit a particularly easy form of cone decomposition [Du 89]. Let $I \subseteq A$ be a Borel-fixed ideal generated by a set of power products of degree $\leq d$, and let B be the set of degree d power products in I. Then I can be expressed as a direct sum (over K) of

1. the power products in I of degree $< d$, and

2. for $i = 1, \ldots, n$, the sets of the form

$$\oplus_{h \in R_i(B)} h \cdot K[x_i, \ldots, x_n] .$$

It then follows that the Hilbert polynomial of I is of the form

$$\overline{\varphi}_I(z) = \sum_{i=1}^{n} |S_i(B)| \binom{z-d+n-i}{n-i} . \tag{1}$$

4 Macaulay's Theorem

Let $L(i, d, m)$ denote the lexicographic set consisting of the m $\underset{L}{>}$-greatest degree d power products of $\mathrm{PP}[x_1, \ldots, x_i]$. Now, for B a Borel fixed set of degree d power products, define

$$T(B) = \cup_{i=1}^{n} \{x_i h : h \in L(i, d-1, |S_i(B)|)\} .$$

Trivially for $i = 1, \ldots, n$,

$$S_i(T(B)) = L(i, d-1, |S_i(B)|) ,$$

and hence

$$|S_i(T(B))| = |S_i(B)| .$$

Lemma 6 *Let B be a Borel-fixed set of degree d power products such that $T(B)$ is also Borel fixed. If*

1. $h = x_1^{a_1} \cdots x_u^{a_u} \in T(B)$ $(a_u \neq 0)$,
2. $g = x_1^{b_1} \cdots x_v^{b_v} \notin T(B)$ $(b_v \neq 0)$, *and*
3. $g \underset{L}{>} h$

then $u < v$.

Proof. By the definition of $T(B)$,

$$h \in T(B) \implies x_u^{-1} h \in L(u, d-1, |S_u(B)|).$$

Now looking for a contradiction, suppose that $u \geq v$. Then

1. $g \underset{L}{>} h \implies x_v^{-1} g \underset{L}{>} x_u^{-1} h$, and
2. $x_v^{-1} g \in \mathrm{PP}[x_1, \ldots, x_u]$

$L(u, d-1, |S_u(B)|)$ is a lexicographic set over $\mathrm{PP}[x_1, \ldots, x_u]$. Since it contains $x_u^{-1} h$, it must also contain the $\underset{L}{>}$-greater power product $x_v^{-1} g$. And so

$$x_v^{-1} g \in L(u, d-1, |S_u(B)|) \implies x_u x_v^{-1} g \in T(B) .$$

Finally, since $T(B)$ is Borel-fixed this leads to the contradiction $g \in T(B)$. □

The usefulness of the set $T(B)$ is that it is *closer* to being a lexicographic ideal than B. This idea of closeness is made explicit in the proof of following lemma.

Lemma 7 *Let $B \subset \mathrm{PP}[x_1, \ldots, x_n]$ be a Borel-fixed set of degree d power products. Suppose that for every Borel-fixed set $C \subset \mathrm{PP}[x_1, \ldots, x_n]$ of degree d power that the set $T(C)$ is also Borel-fixed.*
Then for $k = 1, \ldots, n$,

$$\sum_{i=1}^{k} |S_i(B)| \geq \sum_{i=1}^{k} |S_i(L(n, d, |B|))| .$$

Proof. The assumption concerning sets of the form C is given for technical reasons, and lemma 8 shows that this assumption is always satisfied. For now though, this assumption allows one to conclude that $T(B)$ is Borel-fixed.

If $T(B)$ is a lexicographic set, then since it contains $|B|$ power products of degree d, then it is the lexicographic set $L(n, d, |B|)$. The lemma then follows trivially since for each $i = 1, \ldots, n$,

$$|S_i(B)| = |S_i(T(B))| = |S_i(L(n, d, |B|)| .$$

Assume otherwise that $T(B)$ is not lexicographic. Let $h = x_1^{a_1} \cdots x_u^{a_u}$ be the lexicographically last power product of $T(B)$, and let $g = x_1^{b_1} \cdots x_v^{b_v}$ be the lexicographically first degree d power product not in $T(B) - \{h\}$, where a_u and b_v are non-zero. Since $T(B)$ is not lexicographic $g \underset{L}{>} h$, and by lemma 6, $u < v$.

Let E denote the ideal $(T(B) - \{h\} \cup \{g\})$. Then E is a Borel fixed ideal and for $k = 1, \ldots, n$

$$\sum_{i=1}^{k} |S_i(B)| = \sum_{i=1}^{k} |S_i(T(B))| \geq \sum_{i=1}^{k} |S_i(E)| . \tag{2}$$

Furthermore at $k = u$ the inequality

$$\sum_{i=1}^{u} |S_i(B)| > \sum_{i=1}^{u} |S_i(E)| .$$

is strict and so summing the inequalities (2) for $k = 1, \ldots, n$ yields

$$\sum_{i=1}^{n} i|S_i(B)| > \sum_{i=1}^{n} i|S_i(E)| .$$

Now using induction on the value $\sum_{i=1}^{n} i|S_i(E)|$, one may assume that for $k = 1, \ldots, n$

$$\sum_{i=1}^{k} |S_i(E)| \geq \sum_{i=1}^{k} |S_i(L(n,d,|E|))| .$$

Then,

$$\begin{aligned}\sum_{i=1}^{k} |S_i(B)| &> \sum_{i=1}^{k} |S_i(E)| \\ &\geq \sum_{i=1}^{k} |S_i(L(n,d,|E|))| \\ &= \sum_{i=1}^{k} |S_i(L(n,d,|B|))| .\end{aligned}$$

□

To apply this lemma requires some assurance that the set $T(B)$ will be Borel fixed. The following lemma shows that this is always the case.

Lemma 8 *Let $B \subset \mathrm{PP}[x_1, \ldots, x_n]$ be any Borel fixed set of degree d power products. Then the set $T(B)$ is also Borel fixed.*

Proof. The proof of this lemma uses induction on the number of n and the degree d. The lemma holds trivially for any d if $n = 2$, and it also holds for all n if $d = 1$. So assume inductively that the lemma holds for $\langle d-1, n\rangle$, and $\langle d, n-1\rangle$. To show that $T(B)$ is Borel fixed, consider the three conditions of lemma 5.

1. $T(B) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]$ is Borel fixed.
   Let $C = B \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]$. Then C is a Borel fixed ideal of degree d and $n-1$ variables. By the induction hypothesis $T(C)$ is Borel fixed. But $T(C)$ is exactly the set $T(B) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]$.

2. $S_n(T(B))$ is Borel fixed.
   This is trivial since by construction $S_n(T(B))$ is a lexicographic set.

3. $S_n(T(B)) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}] \subseteq S_{n-1}(T(B))$.
   Since these are both lexicographic sets over degree $d-1$ elements of $\mathrm{PP}[x_1, \ldots, x_{n-1}]$, the inclusion can be proved simply by considering the cardinality of the sets.

   Consider the set $S_n(B)$. This is a Borel-fixed set of n variable power products of degree $d-1$. Using the induction hypothesis, for every set C of this form the set $T(C)$ is also Borel-fixed. And so the previous lemma may be applied with $k = n-1$ to yield,

$$\sum_{i=1}^{n-1} |S_i(S_n(B))| \;\geq\; \sum_{i=1}^{n-1} |S_i(L(n, d-1, |S_n(B)|))| \; .$$

   Furthermore, since B is a Borel set, condition 3 of lemma 5 states that

$$S_n(B) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}] \subseteq S_{n-1}(B)$$

   and so taking cardinalities

$$\sum_{i=1}^{n-1} |S_i(S_n(B))| \;\leq\; |S_{n-1}(B)| \; .$$

   Putting this all together,

$$\begin{aligned}
|S_n(T(B)) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]| &= |L(n, d-1, |S_n(B)|) \cap \mathrm{PP}[x_1, \ldots, x_{n-1}]| \\
&= \sum_{i=1}^{n-1} |S_i(L(n, d-1, |S_n(B)|))| \\
&\leq \sum_{i=1}^{n-1} |S_i(S_n(B))| \\
&\leq |S_{n-1}(B)| \;=\; |S_{n-1}(T(B))| \; .
\end{aligned}$$

□

Lemma 9 *Let I be any Borel fixed ideal generated by k degree d power products, and let L be the lexicographic ideal generated by $L(n,d,k)$. Then for any degree z,*

$$\varphi_I(z) \geq \varphi_L(z) .$$

Proof. For $z < d$, $\varphi_I(z) = \varphi_L(z) = 0$. For $z \geq d$ the Hilbert functions attain the polynomial forms:

$$\begin{aligned} \varphi_I(z) &= \sum_{i=1}^{n-1} |S_i(I)| \binom{z-d+n-i}{n-i} \text{ and} \\ \varphi_L(z) &= \sum_{i=1}^{n-1} |S_i(L(n,d,k))| \binom{z-d+n-i}{n-i} . \end{aligned}$$

So,

$$\varphi_I(z) - \varphi_L(z) = \sum_{i=1}^{n-1} (|S_i(I)| - |S_i(L(n,d,k))|) \binom{z-d+n-i}{n-i} .$$

Using the combinatorial identity $\binom{u+v}{v} = \sum_{j=0}^{v} \binom{u+j-1}{j}$,

$$\begin{aligned} \varphi_I(z) - \varphi_L(z) &= \sum_{i=1}^{n-1} (|S_i(I)| - |S_i(L(n,d,k))|) \left(\sum_{j=0}^{n-i} \binom{z-d+j-1}{j} \right) \\ &= \sum_{j=0}^{n} \left(\binom{z-d+j-1}{j} \sum_{i=1}^{n-j} (|S_i(I)| - |S_i(L(n,d,k))|) \right) . \end{aligned}$$

But from lemma 7, each sum $\sum_{i=1}^{n-j} (|S_i(I)| - |S_i(L(n,d,k))|)$ is a non-negative constant a_j, so

$$\varphi_I(z) - \varphi_L(z) = \sum_{j=0}^{n} a_j \binom{z-d+j-1}{j} \geq 0 .$$

□

Lemma 9 allows an easy proof of Macaualy's theorem.

Theorem 10 (Macaulay) *For every ideal $I \subseteq K[x_1, \ldots, x_n]$ there is a lexicographic ideal L such that I and L have the same Hilbert function.*

Proof. For an arbitrary ideal I, the existence of generic coordinates shows that there is a Borel fixed ideal I such that I and I' share the same Hilbert function. So, we may limit our attention to monomial ideals I which are Borel fixed.

For each degree z, let B_d be the set of degree z power products in I. Since by assumption I is a Borel fixed ideal, each B_z is a Borel fixed set of power products. Now let L be the ideal generated by the infinite set of power products

$$L' = \cup_{z=0}^{\infty} L(n, z, |B_z|) .$$

Now, L must contain only those power products in L'. Otherwise, there would be a degree d and power product p such that $p \in L(n, z, |B_d|)$ but for some x_i, $x_i p \notin L(n, z+1, |B_{d+1}|)$. But this implies

$$\overline{\varphi}_{(B_i)}(i+1) < \overline{\varphi}_{L(n,i,|B_i|)}(i+1) ,$$

contradicting lemma 9.

And so, the set of power products in L is exactly L', and

$$\overline{\varphi}_L(z) = |L(n, z, |B_z|)| = |B_z| = \overline{\varphi}_I(z) .$$

□

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